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X-Splines of Odd Degree

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1. INTRODUCTION

Recently Clenshaw and Negus [3] and Behforooz, Papamichael, and Worsey [2] introduced a number of cubic X-splines in the class of all C^1 piecewise-cubic polynomial interpolatory functions which present several practical advantages in comparison with the conventional cubic splines. A detailed discussion of these advantages can be found in [2, 3, 5, 8]. More recently, Papamichael and Worsey [6] defined a class of C^2 quintic X-splines and investigated convergence and smoothness properties of some X-splines in this class.

In this paper we define X-splines of degree $2n+1$ for a positive integer n. More precisely, let Δ be an arbitrary partition of a compact interval $I=[a, b]$,

$$\Delta\colon a=x_0<x_1<\cdots<x_N=b.$$

Moreover, let us denote

$$h_i=x_i-x_{i-1} \qquad \text{and} \qquad h=\max_i h_i.$$

In Section 2, we study convergence and smoothness properties of C^n piecewise polynomial functions of degree $2n+1$ with breakpoints x_i which interpolate a sufficiently smooth function f at points x_i. The main result of this section consists in establishing a basic property of a piecewise polynomial interpolatory function p which says that the order of the error of approximation of f by p and the orders of jump discontinuities of sth; $s=n+1,..., 2n+1$, derivatives of p at interior knots x_i depend only on the order of approximation of derivatives $f^{(j)}(x_i)$ by $p^{(j)}(x_i)$; $i=0,..., N$, $j=1,..., n$. The results of this section are used in Sections 3 and 4 to define the most general classes of periodic and nonperiodic X-splines of degree $2n+1$, respectively. The definitions of these classes depend on a number of

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free parameters. By specifying these parameters, we define four periodic and nonperiodic X-splines p_{I}–p_{IV} of special interest and present error estimates of approximation of f by these X-splines. The orders of jump discontinuities of sth; $s=n+1,\ldots, 2n+1$, derivatives of the X-splines p_{I}–p_{IV} at points x_i are also presented. Each X-spline of special interest satisfies the following a priori requirements:

(i) It ensures the maximal order $O(h^{2n+2})$ of convergence in the class of all piecewise polynomial interpolatory functions of degree $2n+1$;

(ii) its construction is simpler than the construction of the conventional spline of degree $2n+1$;

(iii) it approximates the jth derivative of f at points x_i with the orders $O(h^{2n+2-j})$; $j=1,\ldots, n$;

(iv) it has a jump discontinuity of the sth derivative of the order $O(h^{2n+2-s})$; $s=n+1,\ldots, 2n+1$.

2. Piecewise Polynomial Interpolation

Let $P_{n,\Delta}$ be the linear space of all C^n piecewise polynomial functions p of degree $2n+1$ or less with breakpoints x_i. Denote $I_i=[x_{i-1}, x_i]$, $t=(x-x_{i-1})/h_i$ and $p_i^{(j)}=p^{(j)}(x_i)$; $j\leqslant n$. Then by the Hermite interpolation formula an element p of $P_{n,\Delta}$ can be expressed in the form

$$p(x)=\sum_{j=0}^{n}\left[p_{i-1}^{(j)}L_{i-1,j}(x)+p_i^{(j)}L_{ij}(x)\right]; \qquad x\in I_i \tag{2.1}$$

where

$$L_{i-1,j}(x)=(h_i^j/j!)(1-t)^{n+1}\sum_{v=0}^{n-j}\binom{n+v}{v}t^{j+v} \tag{2.2}$$

and

$$L_{ij}(x)=(-1)^j L_{i-1,j}(x_{i-1}+x_i-x). \tag{2.3}$$

For a function $f\in C^n(I)$, denote the convex set of all functions $p\in P_{n,\Delta}$ interpolating f at Δ by $P_{n,\Delta}(f)$. An example of a function p in $P_{n,\Delta}(f)$ is the piecewise Hermite function H satisfying the conditions

$$H_i^{(j)}=f^{(j)}(x_i); \qquad i=0,\ldots, N, j=0,\ldots, n.$$

Now suppose in addition that $f\in C^{2n+2}(I_i)$ for each i and denote the linear space of all such f's by $C^n(I;\Delta)$. Then it is well known that the error associated with approximating f by H is of the form

$$f(x)-H(x)=f^{(2n+2)}(\xi)\, w(x)/(2n+2)!; \qquad x\in I_i \tag{2.4}$$

where ξ belongs to the interior I_i^0 of I_i and

$$w(x)=[(x-x_{i-1})(x-x_i)]^{n+1}.$$

Hence we have

$$\|f-H\|\leqslant (h/2)^{2n+2}\|f^{(2n+2)}\|/(2n+2)!. \tag{2.5}$$

In order to estimate $|f(x)-p(x)|$ for $p\in P_{n,\Delta}(f)$, we introduce the notations

$$e_i^{(j)}=f^{(j)}(x_i)-p_i^{(j)}$$

and

$$\lambda_j=\sum_{v=0}^{n-j}\binom{n+v}{v}2^{-(n+j+v)}/j!.$$

Clearly, we have $e_i^{(0)}=0$ and

$$\lambda_j\leqslant (2^{-(n+j)}/j!)\sum_{v=0}^{n}\binom{n+v}{v}2^{-v}=2^{-j}/j!; \qquad j=0,\ldots,n.$$

THEOREM 2.1. If $f\in C^n(I;\Delta)$ *then*

$$|f(x)-p(x)|\leqslant \sum_{j=1}^{n}\lambda_j h_i^j \max\{|e_{i-1}^{(j)}|,|e_i^{(j)}|\} + \frac{h^{2n+2}}{4^{n+1}(2n+2)!}\|f^{(2n+2)}\|; \qquad x\in I_i$$

for a piecewise polynomial function p in $P_{n,\Delta}(f)$.

Proof. By (2.5) it is sufficient to show that the term $|H(x)-p(x)|$ in the inequality

$$|f(x)-p(x)|\leqslant |H(x)-p(x)|+|f(x)-H(x)|$$

can be bounded by the term with λ's occuring in the desired estimate. By virtue of (2.1)–(2.3) we have

$$|H(x)-p(x)|\leqslant \sum_{j=1}^{n} g_j(x)\max\{|e_{i-1}^{(j)}|,|e_i^{(j)}|\}; \qquad x\in I_i$$

where nontrivial and nonnegative polynomials g_j of degree $2n+1$ or less are equal to

$$g_j(x) = |L_{i-1,j}(x)| + |L_{ij}(x)| = L_{i-1,j}(x) + L_{i-1,j}(x_{i-1} + x_i - x).$$

Note that polynomials g_j are symmetric with respect to the midpoint of I_i, i.e., $g_j(x) = g_j(x_{i-1} + x_i - x)$ for every x. This implies that the exact degree d_j of g_j is $2n$ or less. Since $g_j((x_{i-1} + x_i)/2) = \lambda_j h_i^j$ and $g_j(x_{i-1}) = g_j(x_i) = 0$, the proof will be completed if one can show that

$$g_j(z) = \max_{x \in I_i} g_j(x)$$

where $z = (x_{i-1} + x_i)/2$. Suppose this is false. Then by the symmetry of g_j there exist two distinct maxima $z_1, z_2 \in I_i^0 \backslash \{z\}$ of g_j. Thus,

$$g_j'(z_i) = 0; \qquad i = 1, 2. \tag{2.6}$$

On the other hand, the fundamental Hermite polynomial $L_{i-1,j}$ satisfies the conditions

$$L_{i-1,j}^{(s)}(x_{i-1}) = \delta_{js} \quad \text{and} \quad L_{i-1,j}^{(s)}(x_i) = 0; \quad s = 0,..., n.$$

Consequently,

$$g_j^{(s)}(x_{i-1}) = (-1)^s g_j^{(s)}(x_i) = \delta_{js}; \quad s = 0,..., n. \tag{2.7}$$

Now denote $k := \min\{d_j, n\}$. Then a repeated application of Rolle's theorem and relations (2.6)–(2.7) to polynomials $g_j^{(1)},..., g_j^{(k-1)}$ leads us to the conclusion that the polynomial $g_j^{(k)}$ of degree k or less has at least $k+1$ zeros in I_i. Thus $g_j^{(k)} \equiv 0$, which shows that $d_j < k$. This contradiction finishes the proof. ▮

The theorem shows that if the errors $\|e^{(j)}\|_\infty = \max_i e_i^{(j)}$ of approximate values $p_i^{(j)}$ of $f^{(j)}(x_i)$ are such that

$$\|e^{(j)}\|_\infty = O(h^{n_j}); \qquad j = 1,..., n \tag{2.8}$$

then

$$\|f - p\| = O(h^\mu) \tag{2.9}$$

where

$$\mu = \min\{n_1 + 1,..., n_n + n, 2n + 2\}.$$

In particular, the best order $O(h^{2n+2})$ is achieved in (2.9) if

$$n_j \geqslant 2n+2-j \qquad \text{for all } j.$$

Now, we also show that jump discontinuities of the jth derivative of $p \in P_{n,\Delta}(f)$ at interior knots x_i defined by

$$d_i^{(j)}(p) = p^{(j)}(x_i+0) - p^{(j)}(x_i-0); \quad j = n+1, ..., 2n+1$$

depend only on magnitude of $\|e^{(j)}\|_\infty$; $j = 1, ..., n$. For this purpose, we denote

$$\alpha_{sj} = (-1)^{s-n} \frac{s!}{j!} \binom{2n+1-j}{s-j} \binom{s-j-1}{n-j},$$

$$\beta_{sj} = \sum_{m=s}^{2n+1} \alpha_{mj}/(m-s)!, \tag{2.10}$$

$$\gamma_s = 2^{s-n-1}(n+1)!\,(2s-2n-3)!! \binom{s}{2n+2-s} \Big/ (2n+2)!$$

where it is assumed that $(-1)!! = 1$, and start from the following two auxiliary lemmas.

LEMMA 2.1. *If $f \in C^n(I; \Delta)$ then*

$$|f^{(s)}(z) - H^{(s)}(z)| \leqslant \gamma_s h_i^{2n+2-s} \|f^{(2n+2)}\|; \quad s = n+1, ..., 2n+1$$

where $z = x_{i-1}+0,\ x_i - 0$.

Proof. The function

$$g(x) = [f(x) - H(x)] - [f^{(s)}(z) - H^{(s)}(z)]\, w(x)/w^{(s)}(z); \quad x \in I_i$$

has two zeros x_k; $k = i-1, i$ of multiplicity $n+1$. Hence from a repeated application of Rolle's theorem it follows that $g^{(s)}$ has $2n+2-s$ zeros in I_i^0. But we also have $g^{(s)}(z) = 0$. Therefore, by applying Rolle's theorem $2n+2-s$ times to $g^{(s)}, ..., g^{(2n+1)}$ we have $g^{(2n+2)}(\xi) = 0$ for some $\xi \in I_i^0$, which is equivalent to

$$f^{(s)}(z) - H^{(s)}(z) = f^{(2n+2)}(\xi)\, w^{(s)}(z)/(2n+2)!. \tag{2.11}$$

Moreover, in view of the formula (2) from [7, p. 245] we have

$$w^{(s)}(x) = (n+1)!\, h_i^{n+1} \frac{d^{s-n-1}}{dx^{s-n-1}} P_{n+1}\left[2\left(x - \frac{x_{i-1}+x_i}{2}\right) \Big/ h_i \right]; \quad x \in I_i$$

where P_{n+1} is the Legendre polynomial relative to the interval $[-1, 1]$. Hence by the fact that

$$P_{n+1}^{(s-n-1)}(1) = (-1)^s P_{n+1}^{(s-n-1)}(-1) = \binom{s}{2n+2-s} (2s-2n-3)!!$$

we derive

$$w^{(s)}(x_i) = (-1)^s w^{(s)}(x_{i-1}) = 2^{s-n-1}(n+1)! \binom{s}{2n+2-s} \times (2s-2n-3)!!\, h_i^{2n+2-s}. \tag{2.12}$$

Finally, inserting this to (2.11) we directly obtain the desired estimate. ∎

LEMMA 2.2. *The Hermite fundamental polynomials satisfy*

$$L_{i-1,j}^{(s)}(x_{i-1}) = (-1)^{s+j} L_{ij}^{(s)}(x_i) = \alpha_{sj} h_i^{j-s} \tag{2.13}$$

and

$$L_{i-1,j}^{(s)}(x_i) = (-1)^{s+j} L_{ij}^{(s)}(x_{i-1}) = \beta_{sj} h_i^{j-s} \tag{2.14}$$

for all i, j *and* $s = n+1, ..., 2n+1$.

Proof. From (2.3) we immediately obtain the equalities

$$L_{i-1,j}^{(s)}(x_{i-k}) = (-1)^{s+j} L_{ij}^{(s)}(x_{i-m}); \quad k = 0, 1, k+m = 1.$$

In order to prove the remaining equalities we denote

$$g_j(t) = (j!/h_i^j)\, L_{i-1,j}(x)$$

where t is as in (2.2). Now we claim that the coefficient a_{sj} of the polynomial g_j at t^s is equal to

$$a_{sj} = (-1)^{s-n} \binom{2n+1-j}{s-j} \binom{s-j-1}{n-j} \tag{2.15}$$

for $s = n+1, ..., 2n+1$ and $j = 0, ..., n$. Indeed, by making use of the binomial formula to (2.2), we easily find that (2.15) holds for $j = n$ and that g_j can be written in the form

$$g_j(t) = t^{-1} g_{j+1}(t) + \binom{2n-j}{n-j} \sum_{s=0}^{n+1} \binom{n+1}{s} (-t)^s.$$

A comparison of appropriate coefficients on both sides of the last equality gives

$$a_{2n+1,j}=(-1)^{n+1}\binom{2n-j}{n-j}$$

and

$$a_{sj}=a_{s+1,j+1}+(-1)^{j-n}\binom{2n-j}{n-j}\binom{n+1}{s-n};\quad n+1\leqslant s\leqslant 2n.$$

Hence the proof of (2.15) can be easily finished by an induction with respect to s. Moreover, the interpolation conditions $L_{i-1,j}^{(s)}(x_{i-1})=\delta_{sj}$; $s=0,\dots,n$ imply that $a_{sj}=\delta_{sj}$ for $s\leqslant n$. This in conjunction with the definition of g_j gives

$$L_{i-1,j}(x)=(h_i^j/j!)\left[t^j+\sum_{s=n+1}^{2n+1}a_{sj}t^s\right]$$

where a_{sj} are as in (2.15). Hence from the fact that $t=(x-x_{i-1})/h_i$ we deduce that the first terms in (2.13)–(2.14) are equal to the third ones. This completes the proof. ∎

THEOREM 2.2. *If* $f\in C^{2n+2}(I)$ *and* $p\in P_{n,\Delta}(f)$ *then*

$$d_i^{(s)}(p)=\sum_{j=1}^{n}\{\beta_{sj}h_i^{j-s}e_{i-1}^{(j)}-\alpha_{sj}[h_{i+1}^{j-s}-(-1)^{s+j}h_i^{j-s}]\,e_i^{(j)}$$
$$-(-1)^{s+j}\beta_{sj}h_{i+1}^{j-s}e_{i+1}^{(j)}\}+d_i^{(s)}(H)$$

and

$$|d_i^{(s)}(H)|\leqslant 2\gamma_s h^{2n+2-s}\|f^{(2n+2)}\|$$

for all $s=n+1,\dots,2n+1$. *Additionally, if the partition* Δ *is uniform and* $f\in C^{2n+3}(I)$ *then*

$$|d_i^{(s)}(H)|\leqslant 2\gamma_s h^{2n+3-s}\|f^{(2n+3)}\|$$

for each even s.

Proof. From the linearity of $d_i^{(s)}$ it follows that

$$d_i^{(s)}(p)=d_i^{(s)}(p-H)+d_i^{(s)}(H).$$

Moreover, in view of (2.1)–(2.2) and from the fact that

$$p_i^{(j)} - H_i^{(j)} = p_i^{(j)} - f^{(j)}(x_i) = -e_i^{(j)},$$

we have

$$\begin{aligned} d_i^{(s)}(p-H) &= (p-H)^{(s)}(x_i+0) - (p-H)^{(s)}(x_i-0) \\ &= \sum_{j=1}^{n} \{e_{i-1}^{(j)} L_{i-1,j}^{(s)}(x_i) - e_i^{(j)}[(-h_{i+1}/h_i)^{j-s} - 1] \\ &\quad \times L_{ij}^{(s)}(x_i) - e_{i+1}^{(j)} L_{i+1,j}^{(s)}(x_i)\}. \end{aligned}$$

Hence by Lemma 2.2 we derive immediately the desired expression for $d_i^{(s)}(p)$. Since

$$|d_i^{(s)}(H)| \leqslant |H^{(s)}(x_i+0) - f^{(s)}(x_i+0)| + |f^{(s)}(x_i-0) - H^{(s)}(x_i-0)|,$$

we obtain directly from Lemma 2.1 the first estimate for $|d_i^{(s)}(H)|$. Additionally, if Δ is the uniform partition and $f \in C^{2n+3}(I)$ then (2.4) in conjunction with (2.10) and (2.12) implies that

$$\begin{aligned} d_i^{(s)}(H) = H^{(s)}(x_i+0) - H^{(s)}(x_i-0) = \gamma_s h^{2n+2-s}[&f^{(2n+2)}(\xi_{i+1}) \\ &- f^{(2n+2)}(\xi_i)] \end{aligned}$$

for every even s, where $\xi_k \in I_k^0$; $k = i-1, i$. Then, applying the mean value theorem we derive the second estimate for $|d_i^{(s)}(H)|$. ∎

From the last theorem we directly deduce that if the errors $e^{(j)}$ satisfy (2.8) then

$$d_i^{(s)}(p) = O(h^{\mu-s}); \qquad s = n+1, ..., 2n+1 \tag{2.16}$$

where $\mu = \min\{n_1+1, ..., n_n+n, 2n+3\}$ for the uniform partition Δ and μ is as in (2.9) otherwise. In particular, the highest order $O(h^{2n+2-s})$ is achieved here for a partition Δ if

$$n_j \geqslant 2n+2-j \qquad \text{for every } j.$$

3. PERIODIC X-SPLINES

Assume that a function f satisfies the conditions

$$f^{(s)}(a) = f^{(s)}(b); \qquad s = 0, ..., n. \tag{3.1}$$

Moreover, let the partition Δ, the function f and each piecewise polynomial function p in $P_{n,\Delta}$ be extended periodically on the whole real line. Denote by $q_i = q_{ik}$; $k = n+1,..., 2n+1$ the Lagrange interpolating polynomials of degree k or less satisfying the conditions

$$q_i(x_v) = f(x_v) := f_v; \qquad v = i-r,..., i-r+k \tag{3.2}$$

where $r = \text{Entier}\ (k/2)$ and $i = 1,..., N$. Clearly, q_i can be expressed in the form (2.1) and $d_i^{(s)}(q_i) = 0$ for every s. Hence by repeating mutatis mutandis the first part of the proof of Theorem 2.2 we obtain

$$\begin{aligned} d_i^{(s)}(p) &= d_i^{(s)}(p - q_i) \\ &= \sum_{j=1}^{n} \{\beta_{sj} h_i^{j-s} E_{i-1}^{(j)} - \alpha_{sj}[h_{i+1}^{j-s} - (-1)^{s+j} h_i^{j-s}]\ E_i^{(j)} \\ &\quad - (-1)^{s+j} \beta_{sj} h_{i+1}^{j-s} E_{i+1}^{(j)}\}; \qquad s = n+1,..., 2n+1 \end{aligned} \tag{3.3}$$

for each $p \in P_{n,\Delta}(f)$, where

$$E_l^{(j)} = q_i^{(j)}(x_l) - p_l^{(j)}; \qquad l = i-1,\ i,\ i+1.$$

When $n = 1$, then these formulae for $d_i^{(s)}(p)$ reduce to the formulae given recently by Behforooz et al. in [2]. Thus (3.3) can be used to generalize their definition of X-splines. More precisely, let $3Nn^2$ real numbers $a_{ij}^{(s)}$, $b_{ij}^{(s)}$ and $c_{ij}^{(s)}$; $i = 1,..., N$, $j = 1,..., n$, $s = 1,..., n$ be given. Then we define Nn linear functionals $g_i^{(s)}$: $C^n(I) \to \mathbb{R}$ by

$$g_i^{(s)}(y) = \sum_{j=1}^{n} \{a_{ij}^{(s)} y^{(j)}(x_{i-1}) + b_{ij}^{(s)} y^{(j)}(x_i) + c_{ij}^{(s)} y^{(j)}(x_{i+1})\}. \tag{3.4}$$

It is clear from (3.3) that the definition of the functionals $g_i^{(s)}$ is an extension of the defintion of the functionals $d_i^{(n+s)}$; $s = 1,..., n$.

DEFINITION 3.1. A piecewise polynomial function $p \in P_{n,\Delta}(f)$ is called a periodic X-spline of degree $2n+1$ if its parameters satisfy the conditions

$$p_0^{(s)} = p_N^{(s)}; \qquad s = 1,..., n \tag{3.5}$$

and

$$g_i^{(s)}(p) = g_i^{(s)}(q_i); \qquad i = 1,..., N,\ s = 1,..., n. \tag{3.6}$$

EXAMPLE 3.1. Let $p = p_1 \in C^n(I)$ be the periodic X-spline obtained by setting

$$a_{ij}^{(s)} = c_{ij}^{(s)} = 0 \qquad \text{and} \qquad b_{ij}^{(s)} = \delta_{sj}$$

in (3.4) and (3.6). Then its parameters are given explicitly as

$$p_i^{(s)} = q_i^{(s)}(x_i) \qquad \text{and} \qquad p_0^{(s)} = p_N^{(s)}. \tag{3.7}$$

This X-spline is called a diagonal periodic X-spline.

In order to define a periodic X-spline in$C^{n+l}(I)$; $1 \leqslant l \leqslant n$, we may use (3.3). Indeed, a periodic X-spline p belongs to $C^{n+l}(I)$ if and only if

$$d_i^{(s)}(p) = 0; \qquad i = 1,..., N, s = n+1,..., n+l. \tag{3.8}$$

Note that by (2.10) all numbers α_{sj} occuring in (3.3) are integers. Since the numbers α_{mj} are divisible by $(m-j)!$ and $(m-j)!$ is divisible by $(m-s)!$ for every, m, s, j such that $m \geqslant s > j$, we conclude that the numbers β_{sj} occuring in (3.3) are also integers. Moreover, we have

$$\beta_{sj} = (-1)^{n+1} \frac{(n+1)!}{j!} \binom{2n+1-j}{n+1} \Delta^{2n+1-s} g(0)/(2n+1-s)!;$$
$$s = n+1,..., 2n+1, j = 1,..., n \tag{3.9}$$

where the polynomial g of degree n is defined by

$$g(x) = (x+s)(x+s-1) \cdots (x+s-n)/(x+s-j).$$

This formula follows immediately from (2.10) and from the well-known formula for the $(2n+1-s)$th forward difference

$$\Delta^{2n+1-s} g(0) = \sum_{m=0}^{2n+1-s} (-1)^{2n+1-s-m} \binom{2n+1-s}{m} g(m).$$

Since $\Delta^n g(0)/n!$ is equal to the leading coefficient of the polynomial g, the formula (3.9) implies that

$$\beta_{n+1,j} = (-1)^{n+1} \frac{(n+1!}{j!} \binom{2n+1-j}{n+1}; \qquad j = 1,..., n. \tag{3.10}$$

This in conjunction with (3.3)–(3.8) yields the following example of a periodic X-spline in $C^{n+1}(I)$.

EXAMPLE 3.2. Let $p = p_{\text{II}}$ be the periodic X-spline with the coefficients $a_{ij}^{(s)}$, $b_{ij}^{(s)}$ and $c_{ij}^{(s)}$ defined as in Example 3.1 for every $s > 1$. Moreover, let the remaining coefficients be defined by

$$h_i^{n+1-j} a_{ij}^{(1)} = (-1)^{n+j} h_{i+1}^{n+1-j} c_{ij}^{(1)} = \beta_{n+1,j}$$

and

$$b_{ij}^{(1)} = -\alpha_{n+1,j}[h_{i+1}^{j-n-1} + (-1)^{n+j} h_i^{j-n-1}]$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, n$. Then the parameters $p_i^{(s)}$ for $s < n$ are as in (3.7). Furthermore, by inserting the coefficients into (3.4) and (3.6), we conclude from (3.10) that the remaining undefined parameters $p_i^{(n)}$ can be determined from the system of equations

$$a_i p_{i-1}^{(n)} + (-1)^{n+1}(n+1)\, p_i^{(n)} + (1-a_i)\, p_{i+1}^{(n)} = b_i; \qquad i = 1, \ldots, N \tag{3.11}$$

where

$$p_0^{(n)} = p_N^{(n)}, \quad a_i = h_{i+1}/(h_i + h_{i+1}), \quad p_{N+1}^{(n)} = p_1^{(n)},$$

$$\begin{aligned} b_i = {} & a_i q_i^{(n)}(x_{i-1}) + (-1)^{n+1}(n+1)\, q_i^{(n)}(x_i) + (1-a_i)\, q_i^{(n)}(x_{i+1}) \\ & + (-1)^{n+1} \frac{a_i h_i}{n+1} \sum_{j=1}^{n-1} \beta_{n+1,j}[h_i^{j-n-1}(q_i^{(j)} - q_{i-1}^{(j)})(x_{i-1}) \\ & + (-1)^{n+j} h_{i+1}^{j-n-1}(q_i^{(j)} - q_{i+1}^{(j)})(x_{i+1})]. \end{aligned} \tag{3.12}$$

This system is strictly diagonally dominant. Thus the periodic X-spline p_{II} is uniquely determined. Since (3.11) is equivalent to (3.8) for $s = n+1$, it follows that $p_{\mathrm{II}} \in C^{n+1}(I)$.

The polynomials $q_i = q_{ik}$ occurring on the right side of (3.11) depend on k; $n+1 \leqslant k \leqslant 2n+1$. Therefore, there are in fact $n+1$ periodic X-splines of the type p_{II} defined by (3.11) with the right sides $b_i = b_{ik}$ dependent on k. In particular, when $n = 1$ then

$$b_{i2} = b_{i3} = 3a_i[f_{i-1}, f_i] + 3(1-a_i)[f_i, f_{i+1}]$$

where $[f_{j-1}, f_j] = (f_j - f_{j-1})/h_j$ is a divided difference of order 1. Thus the definition of p_{II} for $n = 1$ is independent of $k = 2, 3$ and p_{II} coincides with the well-known conventional periodic cubic spline.

Now we proceed to investigate convergence properties of periodic X-splines. For this purpose, suppose that $r_m = r_{m,k,i}$ is the remainder term of the Lagrange interpolation formula of degree k with knots x_v; $v = i-r, \ldots, i-r+k$ for the function $(x - x_i)^m$; $k < m \leqslant 2n+1$. Then

$$r_m(x) = (x - x_i)^m - g(x)$$

where the polynomial g of degree k or less is uniquely determined by the conditions

$$g(x_v) = (x_v - x_i)^m; \qquad v = i-r, \ldots, i-r+k$$

with r as in (3.2). Additionally, let $R = R_{ki}$ be the remainder term of the same interpolation formula for the function

$$\hat{f}(x) = f(x) - \sum_{m=k+1}^{2n+1} f^{(m)}(x_i)(x - x_i)^m/m!.$$

Then by the linearity of a remainder in Lagrange interpolation together with the linearity of $g_i^{(s)}$ we have

$$g_i^{(s)}(f - q_i) = \sum_{m=k+1}^{2n+1} f^{(m)}(x_i)\, g_i^{(s)}(r_m)/m! + g_i^{(s)}(R) \tag{3.13}$$

for all $f \in C^{2n+1}(I)$. This formula will play a fundamental role in error analysis for periodic X-splines, since it gives a useful expansion of the right sides of the following equalities equivalent to (3.6):

$$g_i^{(s)}(f - p) = g_i^{(s)}(f - q_i); \qquad i = 1, ..., N,\ s = 1, ..., n. \tag{3.14}$$

From the definition of $g_i^{(s)}$ we easily note that the right side of (3.13) is a linear combination of quantities $r_m^{(j)}(x_\mu)$ and $R^{(j)}(x_\mu)$, where $m = k+1, ..., 2n+1$, $j = 1, ..., n$ and $\mu = i-1, i, i+1$. Since R is a remainder of the Lagrange interpolation formula, it follows from Theorem 1 in Section 6.5 of [4] that

$$R^{(j)}(x_\mu) = \frac{\hat{f}^{(k+1)}(\eta)}{(k+1-j)!} \prod_{v=0}^{k-j} (x_\mu - \xi_v) \tag{3.15}$$

where

$$x_{i-r} < \eta < x_{i-r+k} \quad \text{and} \quad x_{i-r+v} < \xi_v < x_{i-r+v+j}.$$

Hence by Taylor's series expansion of $\hat{f}^{(k+1)}(\eta)$ at the point x_i and by $\hat{f}^{(2n+2)} = f^{(2n+2)}$ and $\hat{f}^{(j)}(x_i) = 0$ for $j = k+1, ..., 2n+1$ we have

$$R^{(j)}(x_\mu) = \frac{f^{(2n+2)}(\sigma)(\eta - x_i)^{2n+1-k}}{(k+1-j)!(2n+1-k)!} \prod_{v=0}^{k-j} (x_\mu - \xi_v) \tag{3.16}$$

for $f \in C^{2n+2}(\mathbb{R})$, where $x_{i-r} < \sigma < x_{i-r+k}$. Consequently, we obtain

$$|R^{(j)}(x_\mu)| \leqslant C_j h^{2n+2-j} \|f^{(2n+2)}\| \tag{3.17}$$

with a constant C_j independent of h and f. Moreover, this constant can be estimated as

$$C_j \leqslant \frac{k^{2n+2-j}}{(k+1-j)!(2n+1-k)!}.$$

Since r_m is also a remainder of the Lagrange interpolation formula with the same knots as R, it follows that $r_m^{(j)}(x_\mu)$ can be expressed in the form (3.15) with f replaced by $(\cdot - x_i)^m$. From this we conclude that

$$|r_m^{(j)}(x_\mu)| \leqslant D_j h^{m-j} \tag{3.18}$$

where the constant D_j independent of h can be estimated as

$$D_j \leqslant k^{m-j}/(k+1-j)!.$$

We note that the upper bounds for C_j and D_j are the simplest and at the same time the largest ones. Since these bounds are sufficient for our purposes, we do not have to worry about decreasing them. Now we can establish the following results concerning convergence and smoothness properties of periodic X-splines p_{I} and p_{II}.

THEOREM 3.1. *Let $p = p_{\mathrm{I}}$, p_{II} be periodic X-splines for $k = 2n+1$ interpolating a C^{2n+2}-periodic function f with the period $b-a$. Then we have*

$$\|f - p\| \leqslant C h^{2n+2} \|f^{(2n+2)}\| \tag{3.19}$$

with a constant C depending only on n. Additionally,

$$d_i^{(s)}(p) = O(h^{2n+2-s}); \qquad s = n+1, ..., 2n+1 \tag{3.20}$$

where it is assumed that $s > n+1$ for $p = p_{\mathrm{II}}$.

Proof. If $p = p_{\mathrm{I}}$, then

$$q_i^{(j)}(y) = y^{(j)}(x_i); \qquad j = 1, ..., n.$$

Since $k = 2n+1$, it follows from (3.13)–(3.14) and (3.17) that

$$|e_i^{(j)}| = |R^{(j)}(x_i)| \leqslant C_j h^{2n+2-j} \|f^{(2n+2)}\|; \quad j = 1, ..., n. \tag{3.21}$$

Hence by Theorem 2.1 we obtain

$$|f(x) - p(x)| \leqslant h^{2n+2} \|f^{(2n+2)}\| \left[\sum_{j=1}^{n} \lambda_j C_j + 4^{-n-1}/(2n+2)! \right]; \ x \in I_i,$$

which completes the proof of (3.19) for $p = p_{\mathrm{I}}$. Further, from (3.21) and Theorem 2.2 we immediately conclude that (3.20) holds in this case. Now, suppose that $p = p_{\mathrm{II}}$ and $k = 2n+1$. Then $|e_i^{(j)}|; j = 1, ..., n-1$ have the same estimates as in (3.21). Thus, in view of Theorems 2.1 and 2.2, it is sufficient to show that (3.21) holds also for $j=n$. For this purpose, let us note first that the relations (3.14) can be written for $s=1$ in the form

$$a_i e_{i-1}^{(n)} + (-1)^{n+1}(n+1)\, e_i^{(n)} + (1-a_i)\, e_{i+1}^{(n)} = c_i; \quad i=1,..., N$$

with $e_0^{(n)} = e_N^{(n)}$, where the a_i are as in (3.12) and the c_i are equal to b_i defined in (3.12) with q_μ; $\mu = i-1,\ i,\ i+1$ replaced by $f - q_\mu$. Hence the standard considerations (see, e.g., [1, p. 24]) lead to the conclusion

$$|e_l^{(n)}| \leqslant n^{-1} \max_{1 \leqslant i \leqslant N} |c_i|; \qquad l=1,..., N. \tag{3.22}$$

But the c_i are linear combinations of the quantities $(f-q_i)^{(j)}(x_\mu)$ and $(f-q_v)^{(j)}(x_v)$; $\mu = i-1,\ i,\ i+1,\ v = i-1,\ i+1$, which by virtue of (3.13)–(3.14) and (3.17), have the same estimates as $e_i^{(j)}$ in (3.21). This and (3.22) imply that

$$\begin{aligned} |e_l^{(n)}| &\leqslant n^{-1} h^{n+2} \|f^{(2n+2)}\| \left[(n+2)\, C_n + 4 \sum_{j=1}^{n-1} |\beta_{n+1,j}| C_j/(n+1) \right] \\ &:= C_n h^{n+2} \|f^{(2n+2)}\| \end{aligned}$$

where $\beta_{n+1,j}$ are as in (3.10). Thus (3.21) holds for $j=n$ and the proof is completed. ▮

It is important to note that Theorem 3.1 is false for $k \leqslant 2n$. This is an immediate consequence of (3.18), which implies that the estimates of the right sides of (3.14) depend on the quantities $g_i^{(s)}(r_m)$; $k+1 \leqslant m \leqslant 2n+1$ of order less than $2n+2$. We may partially avoid these difficulties by introducing a new class of X-splines with coefficients $a_{ij}^{(s)}$, $b_{ij}^{(s)}$ and $c_{ij}^{(s)}$ satisfying the conditions

$$g_i^{(s)}(r_m) = 0; \qquad m = k+1,..., 2n+1. \tag{3.23}$$

Now we discuss the simplest case of such X-splines obtained for $k=2n$.

EXAMPLE 3.3. Let $p = p_{\mathrm{III}} \in C^n(I)$ be the periodic X-spline obtained by setting

$$q_i = q_{i,2n}, \qquad a_{ij}^{(s)} = 0, \qquad b_{ij}^{(s)} = \delta_{js}, \qquad c_{ij}^{(s)} = \delta_{js} c_{is}$$

in (3.4) and (3.6), where the coefficients c_{is} are such that (3.23) holds. Then we have

$$g_i^{(s)}(y) = y^{(s)}(x_i) + c_{is} y^{(s)}(x_{i+1}). \tag{3.24}$$

Consequently, the conditions (3.5)–(3.6) give n systems of equations of the form

$$p_i^{(s)} + c_{is} p_{i+1}^{(s)} = q_i^{(s)}(x_i) + c_{is} q_i^{(s)}(x_{i+1}); \; i = 1,..., N, \; s = 1,..., n \tag{3.25}$$

for determining $p_i^{(s)}$, where $p_{N+1}^{(s)} = p_1^{(s)}$. In addition, (3.23) implies that

$$c_{is} = -r_{2n+1}^{(s)}(x_i)/r_{2n+1}^{(s)}(x_{i+1}). \tag{3.26}$$

By the Lagrange interpolation formula the remainder r_{2n+1} can be expressed in the form

$$r_{2n+1}(x) = r_{2n+1,i}(x) = \prod_{m=i-n}^{i+n} (x - x_m). \tag{3.27}$$

In particular, in the case $n = 1$, we have

$$c_{i1} = h_i/(h_i + h_{i+1}); \qquad i = 1,..., N.$$

Therefore, the system (3.25) is strictly diagonally dominant, which implies the existence and uniqueness of the cubic X-spline p_{III} ($n = 1$). Further, in case $n = 2$, we obtain from (3.26)–(3.27) the formula

$$c_{i1} = \frac{h_i(h_i + h_{i-1})(h_{i+2} + h_{i+1})}{(h_{i+1} + h_i + h_{i-1})(h_{i+1} + h_i)\, h_{i+2}}.$$

This shows that the systems in (3.25) are not strictly diagonally dominant in general. However, when the partition Δ is uniform, then $c_{i1} = 2/3$ for $n = 2$. Consequently, in this case the first system in (3.25) is strictly diagonally dominant. Now, suppose that the partition Δ is uniform and that n is a positive integer. Then the sth derivative of r_{2n+1} at x_i divided by $s!$ is an elementary symmetric function [9] of degree $\sigma = 2n + 1 - s$ in the arguments $u_m = (x_i - x_{i-m}) = mh$; $m = \pm 1,..., \pm n$. More precisely, we have

$$r_{2n+1}^{(s)}(x_i) = s! \sum u_{k_1} \cdots u_{k_\sigma}$$

where the sum is extended to every combination of order σ of the numbers $-n$, $-n+1,...,$ -1, $1, 2,..., n$ without repetition and without permutation. Since $u_{-m} = -u_m$, it follows that

$$r_{2n+1}^{(s)}(x_i) = 0; \qquad s = 2, 4,... \tag{3.28}$$

and

$$r_{2n+1}^{(s)}(x_i) = (-1)^{\sigma/2} s! h^\sigma M_{2n+1,s}; \qquad s = 1, 3,... \tag{3.29}$$

where

$$M_{2n+1,s} = \sum (k_1 \cdots k_{\sigma/2})^2. \tag{3.30}$$

Here the sum is extended to every combination of order $\sigma/2$ of the numbers $1,..., n$ without repetition and without permutation. As an immediate consequence of (3.30) we get

$$M_{2n+1,s} = M_{2n-1,s-2} + n^2 M_{2n-1,s}. \tag{3.31}$$

Starting from the initial conditions

$$M_{2n+1,1} = (n!)^2 \quad \text{and} \quad M_{2n+1,2n+1} = 1, \tag{3.32}$$

which follow directly from (3.30), we may compute step by step the positive numbers $M_{2n+1,s}$ with the aid of formula (3.31) (Table I). Next, let us apply Leibnitz formula for the sth derivative of a product to the particular product

$$r_{2n+1,i}(x) = [(x - x_{i-n})(x - x_{i-n+1})]\, r_{2n-1,i+1}(x).$$

Then, in view of (3.28)–(3.30), it can be found that

$$r^{(s)}_{2n+1,i}(x_{i+1}) = (-1)^{(\sigma-1)/2} s(2n+1)\, s!h^\sigma M_{2n-1,s-1} \neq 0; \;\; s = 2, 4,...$$

and

$$r^{(s)}_{2n+1,i}(x_{i+1}) = (-1)^{(\sigma-2)/2} s!h^\sigma [n(n+1)\, M_{2n-1,s} - M_{2n-1,s-2}]; \; s = 1, 3,...$$

where it is additionally assumed that $M_{2n-1,-1} = 0$ and $M_{11} = 1$. Hence by

TABLE I

Table of the Numbers $M_{2n+1,s}$

$n \backslash s$	1	3	5	7
1	1	1		
2	4	5	1	
3	36	49	14	1
4	576	820	273	30
5	14400	21076	7645	1023
6	518400	773136	296296	44473
7	25401600	38402064	15291640	2475473
8	1625702400	2483133696	1017067024	173721912

(3.26) in conjunction with (3.28)–(3.29) and (3.32) we deduce that the coefficients c_{is}; $s=1,\dots,n$ in (3.25) are independent of i and equal to

$$c_{is}=0; \qquad s=2,4,\dots \tag{3.33}$$

and

$$c_{is}=\frac{M_{2n+1,s}}{(n+1)nM_{2n-1,s}-M_{2n-1,s-2}}; \quad s=1,3,\dots. \tag{3.34}$$

By aid of the last formula we may easily compute a table of the nonzero coefficients $c_{is}=c_{isn}$ (Table II). Finally, we note that construction of the periodic X-spline p_{III} of degree $2n+1$ corresponding to the uniform partition involves solving Entier$((n+1)/2)$ systems of equations of the type (3.25) which are strictly diagonally dominant for X-splines of degree less than 11 and equal to 13.

EXAMPLE 3.4. Let $p=p_{\mathrm{IV}}\in C^n(I)$ be a periodic X-spline corresponding to the following choice of its coefficients

$$a_{ij}^{(s)}=\delta_{js}a_{is}, \qquad b_{ij}^{(s)}=\delta_{js}, \qquad c_{ij}^{(s)}=0$$

where a_{is} satisfies (3.23) for $k=2n$. Then the parameters $p_i^{(s)}$ satisfy n systems of equations of the form

$$a_{is}p_{i-1}^{(s)}+p_i^{(s)}=a_{is}q_i^{(s)}(x_{i-1})+q_i^{(s)}(x_i); \quad i=1,\dots,N,\ s=1,\dots,n$$

where $p_0^{(s)}=p_N^{(s)}$. Moreover, we have

$$a_{is}=-r_{2n+1}^{(s)}(x_i)/r_{2n+1}^{(s)}(x_{i-1})$$

TABLE II

Table of the Coefficients c_{is}

n \ s	1	3	5	7
1	1/2			
2	2/3			
3	3/4	7/8		
4	4/5	205/236		
5	5/6	479/546	139/134	
6	6/7	2478/2791	1036/1049	
7	7/8	$\frac{266681}{297064}$	4201/4346	2473/2199
8	8/9	$\frac{1141146}{1258983}$	$\frac{3739217}{3906603}$	14037/13166

where r_{2n+1} is as in (3.27). Hence it is clear that all results from the previous example remain valid under the additional assumption that c_{is}, h_{i+1} and x_{i+1} are replaced by a_{is}, h_{i-1} and x_{i-1}, respectively. In particular, if the knots x_i are uniformly spaced then $a_{is} = c_{is}$.

THEOREM 3.2. *Let f be a C^{2n+2}-periodic function with the period $b-a$ and let $p = p_{\text{III}}$, p_{IV} be periodic X-splines of degree $2n+1$; $n = 1, 2, 3, 4, 6$ interpolating f at uniformly spaced knots x_i. Then we have*

$$\|f - p\| \leqslant Ch^{2n+2}\|f^{(2n+2)}\|$$

and

$$d_i^{(s)}(p) = O(h^{2n+2-s}); \qquad s = n+1,..., 2n+1$$

with a constant C depending on n only.

Proof. We prove the results for $p = p_{\text{III}}$ only, since the proof for p_{IV} is the same. For this purpose, denote by A_j the matrix of the jth system in (3.25). This matrix is strictly diagonally dominant and its elements c_{ij}; $j = 1,..., n$ are independent of i. Therefore, we have

$$\|A_j^{-1}\|_\infty \leqslant (1 - c_{ij})^{-1}.$$

Hence by using (3.13)–(3.14) and (3.23)–(3.24) we obtain

$$\begin{aligned} |e_i^{(j)}| &\leqslant (1 - c_{ij})^{-1} \max_{1 \leqslant i \leqslant N} |g_i^{(j)}(R)| \\ &= \max\{|R_{2n,i}^{(j)}(x_\mu)| : i = 1,..., N, \mu = i, i+1\}. \end{aligned}$$

This, in view of (3.17), implies that $|e_i^{(j)}|$ have estimates of the form (3.21). Finally, inserting these estimates into Theorems 2.1 and 2.2 we immediately obtain the desired results. ▮

It would be interesting to determine all s's such that the sth system in (3.25) is not strictly diagonally dominant in the case of uniformly spaced knots x_i for an integer $n > 8$. Since by (3.34) we have $c_{i1} = n/(n+1) < 1$, it follows from (3.31)–(3.34) that this problem will be solved if we determine all odd integers s; $1 < s \leqslant n$ such that the following inequality holds:

$$nM_{2n-1,s} \leqslant 2M_{2n-1,s-2}.$$

This is left as an open problem. Moreover, it is obvious that Definition 3.1 may be used to define a number of other particular periodic X-splines given above and preserving the highest order of convergence. For example, we can define the X-spline $p = p_V \in C^{n+1}(I)$ with parameters $p_i^{(s)}$ defined as a solution of systems (3.25) and (3.11) for $s = 1, \dots, n-1$ and $s = n$, respectively. Note that in this case Theorem 3.2 holds at least for $n \leqslant 8$.

4. Nonperiodic X-Splines

Now we define nonperiodic X-splines for a function f such that at least one condition in (3.1) is not satisfied. In this case, we change the definition of polynomials $q_i = q_{ik}$; $i = 1, \dots, r-1$, $N+r-k+1, \dots, N-1$ as follows:

$$\begin{aligned} q_1 = \cdots &= q_{r-1} := q_r, \\ q_{N-1} = \cdots &= q_{N+r-k+1} := q_{N+r-k}. \end{aligned} \tag{4.1}$$

Analogously, we change the definition of $r_m = r_{m,k,i}$ and $R = R_{ki}$; $1 \leqslant i < N$ for the first $r-1$ and the last $k-r-1$ values of i.

Definition 4.1. A piecewise polynomial function $p \in P_{n,\Delta}(f)$ is called a nonperiodic X-spline of degree $2n+1$ if its parameters $p_i^{(s)}$ satisfy the conditions

$$p_i^{(s)} = f_i^{(s)}; \qquad i = 0, N, s = 1, \dots, n \tag{4.2}$$

and

$$g_i^{(s)}(p) = g_i^{(s)}(q_i); \qquad i = 1, \dots, N-1, s = 1, \dots, n \tag{4.3}$$

where $g_i^{(s)}$ are as in (3.4).

If we set $n = 1$ in the above definition then in accordance with (2.10) and (3.3)–(3.4), we obtain the definition of cubic X-splines stated in [2, 3]. Similarly, if we insert

$$n = 2, \ q_i = q_{i,3}, \ a_{ij}^{(s)} = \delta_{js} a_{is}, \ b_{ij}^{(s)} = \delta_{js}, \ c_{ij}^{(s)} = \delta_{js} c_{is}$$

into Definition 4.1, then we obtain the definition of quintic X-splines which were given recently in [6]. Moreover, by making use of Definition 4.1, we can define the nonperiodic X-splines p_{I}–p_{II} of degree $2n+1$ in a way similar to that shown in the preceding section. We omit details here, since they involve minor changes such as removing periodic end conditions and Nth equations from the systems of equations defining the periodic X-splines p_{I} and p_{II} and joining the equalities $p_0^{(s)} = f_0^{(s)}$ and $p_N^{(s)} = f_N^{(s)}$ at the beginning and the end of the sth system, respectively. Furthermore, by repeating mutatis mutandis the considerations from Section 3, we conclude that Theorem 3.1 holds for the nonperiodic X-splines p_{I} and p_{II} interpolating a function $f \in C^{2n+2}(I)$. The values $f_i^{(s)}$; $i=0, N$ are not usually available in practice. However, we may replace them by suitable approximations setting

$$p_i^{(s)} = q_{i,2n+1}^{(s)}(x_i); \qquad i=0, N,\ s=1,\ldots, n. \tag{4.2'}$$

Then reasoning similar to that in the proof of (3.21) leads us to the conclusion that if $f \in C^{2n+2}(I)$ then

$$e_i^{(j)} = O(h^{2n+2-j}); \qquad i=0, N,\ s=1,\ldots, n.$$

Therefore, the choice of the end coefficients $p_i^{(s)}$ preserves the highest order of convergence of the nonperiodic X-splines p_{I} and p_{II} to f. This process can be extended to define the nonperiodic X-splines p_{III} and p_{IV} with the coefficients $p_i^{(s)}$; $i=n,\ldots, N-n$ as given in Examples 3.3 and 3.4 and with the coefficients $p_i^{(s)}$; $i=1,\ldots, n-1,\ N-n+1,\ldots, N-1$ defined by

$$p_i^{(s)} = q_{i,2n+1}^{(s)}(x_i).$$

The end parameters $p_i^{(s)}$; $i=0, N$ can be selected here as in (4.2) or (4.2′). Obviously, if $f \in C^{2n+2}(I)$ then Theorem 3.2 holds for these X-splines.

Finally, we note that the method of proving Theorems 3.1 and 3.2 makes it possible to compute the values of the constants in the estimates occurring there. Clearly, this calculation can be done effectively for a few small values of n only. In particular, such estimates will be given in our next paper for a number of quintic periodic and nonperiodic X-splines.

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